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Ectoplasm Has No Topology: The Prelude[1, 2]

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ABSTRACT

Preliminary evidence is presented that a long overlooked and critical element in the fundamental definition of a general theory of integration over curved Wess-Zumino superspace lies with the imposition of "the Ethereal Conjecture" which states the necessity of the superspace to be topologically "close" to its purely bosonic sub-manifold. As a step in proving this, a new theory of integration of closed super p-forms is proposed.

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Presently ‘Salam-Strathdee superspace [1]’ is almost universally accepted as the requisite mathematical setting for describing supersymmetrical field theories. Even so, there remain a fairly large number of open questions about superspace, particularly with regard to those with large values of N ($\equiv N_F$) or D ($\equiv N_B$). There are also particularly pointed questions that remain largely unanswer about a general theory of integration on the curved versions of these spaces also known as ‘Wess-Zumino superspace [2].’ To answer some of these questions, extensions such as ‘harmonic superspace’ have been developed especially by the late Dr. Ogievetsky and collaborators. Although my discussion today will only tangentially touch on such constructions, I wish to dedicate this talk to Victor Isaakovich’s memory.

Near the beginning of research using superspace, more mathematically motivated investigators such as Rogers [3] asked a question we may paraphrase as,

> “Is it possible to construct a superspace whose topological properties are significantly different from those of its purely bosonic subspace?”

In all cases of interest to physicists to date the answer appears to be, “No!” The emphasis on this negation is mine own because I believe that there is a hidden message in this answer.

In establishing a nomenclature appropriate to researching these issues, one often finds the ‘spiritualist’ denotations (see for example [4])

monomials in $x \equiv$ body of the superspace,

monomials in x and θ or purely $\theta \equiv$ soul of the superspace.

In deference to this convention, I may call the ‘basic substance’ of which the soul is composed, the “ectoplasm” of superspace.

There is a peculiar sense in which the question of how to construct integration measures over <u>curved</u> superspaces is unanswered. Arnowitt, Nath and Zumino [5] first suggested such integration measures should be written as

$$\int d\mu ~\equiv~ \int d^{N_B+N_F}z ~\mathrm{E}^{-1} ~=~ \int d^{N_B+N_F}z ~[sdet\,(\mathrm{E}_{\underline{A}}{}^{\underline{M}}(\theta,x)\,)]^{-1} \quad . \tag{1}$$

for a superspace of N_B bosonic coordinate and N_F fermionic coordinates.

In principle this is perfectly consistent. In practice, however, for any theory with large N_B or N_F ($N_F = 4$ is large), this becomes an impractical way to obtain component results in a supergravity theory of ‘physical’ interest. The impracticality arises because the complete θ-expansion of the superdeterminant of the inverse vielbein

$[sdet\,(\mathrm{E}_{\underline{A}}{}^{\underline{M}}(\theta,x)\,)]^{-1}$ is complicated to calculate[4]. For practical calculations an alternative to the method of Arnowitt, Nath and Zumino is required. To my knowledge, only two such alternatives exist in the literature. They have been discussed in three books listed by authors below.

a. "Covariant Theta Expansion" - Wess & Bagger, [6]

b. "Density Projectors" - Gates, Grisaru, Roček & Siegel, [7]
- Buchbinder & Kuzenko. [8]

I will obviously speak on the second of these because I have recently found increasing and unexpected indications that it is directly connected to more general issues of the calculus and topology of curved supermanifolds with torsion.

I begin by writing the "Ectoplasmic Integration Theorem" (or E.I.T.). There should exist an operator $\mathcal{D}^{N_F}$ such that

$$\int d^{N_B+N_F}z\;\mathrm{E}^{-1}\mathcal{L} \;=\; \int d^{N_B}z\;\mathrm{e}^{-1}\,[\,\mathcal{D}^{N_F}\mathcal{L}\,|\,] \quad , \tag{2}$$

independent of the superfield $\mathcal{L}$ that appears in this equation and where

$$\mathrm{e}^{-1} \;\equiv\; [det\,(\mathrm{e}_{\underline{a}}{}^{\underline{m}}(x)\,)]^{-1} \quad , \quad \mathcal{D}^{N_F}\mathcal{L}| \;\equiv\; \lim_{\theta\to 0}\,(\,\mathcal{D}^{N_F}\mathcal{L}) \quad . \tag{3}$$

This theorem is of a similar form to that of the standard Gauss', Green's or Stoke's Theorems of multi-variable calculus. It is different, however, because the operator $\mathcal{D}^{N_F}$ appears on the "wrong" side of the equation from the standard multi-variable calculus analogs. The E.I.T. is also the natural extension of the Berezinian definition of integrating over Grassmann numbers [9].

To see why this is a practical improvement in calculational matters, let me consider the case of flat 4D, N = 1 superspace where the E.I.T. becomes

$$\int d^4x\,d^2\theta\,d^2\bar{\theta}\;\mathcal{L} \;\equiv\; \tfrac{1}{2}\Big\{\int d^4x\;[\,D^2\,\overline{D}{}^2\,\mathcal{L}\,|\,] \;+\; \mathrm{h.c.}\Big\} \quad , \tag{4}$$

where

$$D_\alpha \;\equiv\; \partial_\alpha \;+\; i\tfrac{1}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\underline{a}} \quad , \quad \overline{D}_{\dot{\alpha}} \;\equiv\; \bar{\partial}_{\dot{\alpha}} \;+\; i\tfrac{1}{2}\theta^{\alpha}\partial_{\underline{a}} \quad . \tag{5}$$

Anyone familiar with rigid supersymmetry can attest to the practical utility of the above equation. For example, if I define $\mathcal{L} \;\equiv\; \overline{\Phi}\Phi$ where $\overline{D}_{\dot{\alpha}}\Phi \;=\; 0$ use the

[4]To my knowledge, this calculation has only been done explicitly by no more than six physicists to this date for 4D, N = 1 supergravity.

component field definitions $A(x) \equiv \Phi|$, $\psi_\alpha(x) \equiv D_\alpha \Phi|$ and $F(x) \equiv D^2 \Phi|$, apply the E.I.T. and use of the Leibnitz rule for differentiation, it is simple to show

$$\int d^4 x \, d^2\theta \, d^2\bar\theta \; \overline{\Phi}\Phi \;=\; \int d^4 \, x \, [\; -\tfrac{1}{2}(\partial^{\underline{a}}\overline{A})\,(\partial_{\underline{a}}A) \;-\; i\overline{\psi}^{\dot\alpha}\partial_{\underline{a}}\psi^\alpha \;+\; F\overline{F} \;] \quad . \tag{6}$$

No explicit θ-expansion was required at any point to derive this component result. Thus, it should be obvious why it is calculationally superior to use the E.I.T. By using techniques that are essentially the same as above, we simple by-pass the need to know the explicit structure of the θ-expansion of $[sdet\,(\mathrm{E}_{\underline{A}}{}^{\underline{M}}(\theta, x)\,)]^{-1}$!

From this viewpoint, the whole problem becomes how to develop a theory for the calculation of the operator $\mathcal{D}^{N_F}$ that appears in equation (2). The expression $e^{-1}[\mathcal{D}^{N_F}\mathcal{L}|\,]$ is called "the density projection operator" or "density projector" (see '*Superspace*' [10] or '*Ideas*' [11]). It should be clear that this operator, in the general case, can be written as

$$\int d^{N_B} z \; \mathrm{e}^{-1} \Big[\; \mathcal{D}^{N_F}\mathcal{L}\,| \;\Big] \;=\; \int d^{N_B} z \; \mathrm{e}^{-1} \Big[\; \sum_{i=0}^{N_F} c_{(N_F-i)}\,(\nabla \;\cdot\;\cdot\;\cdot\; \nabla)^{N_F-i}\,\mathcal{L}| \;\Big] \quad , \tag{7}$$

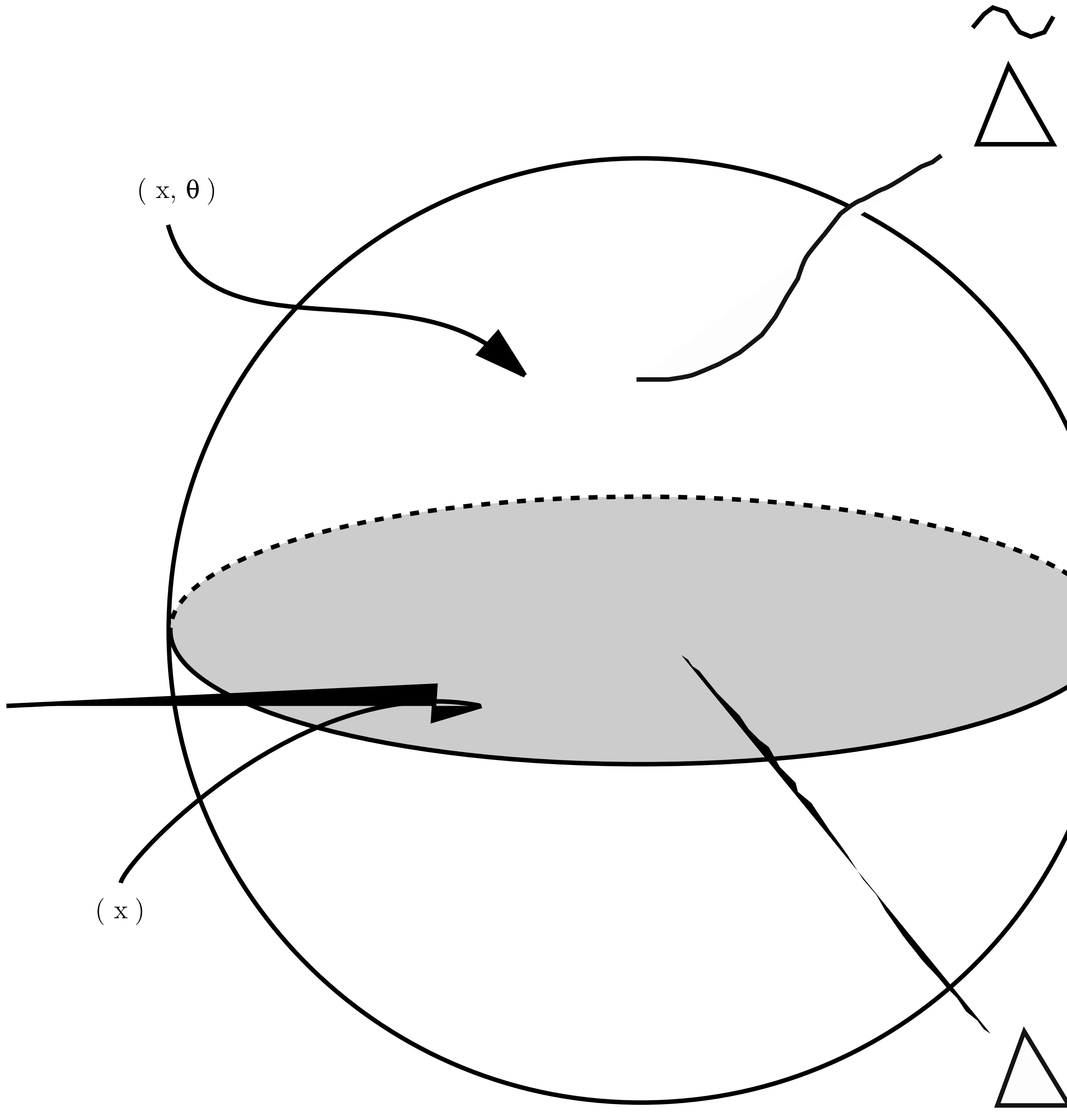
in terms of some field-dependent coefficients $c_{(N_F-i)}$ and powers of the spinorial superspace supergravity covariant derivative ∇_α. How are these coefficients to be found?

In '*Superspace*' [10] it was shown that given the local supersymmetry variations of some matter superfield, it is possible to re-construct these coefficients. In '*Ideas*' [11], it was shown that the density projector follows after solving the constraints to find the basic supergravity pre-potentials. Neither of these approaches is a theory[5] for $\mathcal{D}^{N_F}$. In the early to middle eighties, Zumino was the first to raise the question of a purely *theoretical* basis for this operator. This bring us to the point of my presentation.

In the rest of my presentation, I will attempt to convince the reader that the answer can be found in the study of super topology similar to the investigations by Rogers. I will argue that local supergravity theories (as a principle) obey what I call "The Ethereal Conjecture" which largely determines the form of $\mathcal{D}^{N_F}$.

[5]We may think of the approach in [10] as a 'handicraft' method for summarizing component results. The fact that it was required to go component at all, was equivalent to an admission that we did not have an *a priori* theoretical basis for this result.

A Representation of Superspace



The volume of the sphere represents the entirety of superspace and t he equatorial plane represents the bosonic sub-space.

In terms of the topological indices represented in the diagram I may formulate the Ethereal Conjecture (E. C.) as:

For all Wess-Zumino superspaces, the operator $\mathcal{D}^{N_F}$ that appears in the E. I .T. has the property that it insures that $\widehat{\Delta} \simeq \Delta$.

I will later give some meaning to $\simeq$ that appears in this relation. This is the result to which I was alluding in the other-worldly sounding title. Stated another way, this relation says that the local integration measure of a Wess-Zumino superspace is such that the superspace is topologically "close" to its underlying bosonic manifold. Or alternately, the local Grassmann integration measure of a Wess-Zumino superspace is actually to be detemined from topological considerations.

For an ordinary N_B-form $f_{\underline{a}_1 \cdots \underline{a}_{N_B}}$ defined over an ordinary bosonic manifold, the calculation of the index Δ is just given by

$$\Delta \equiv (N_B!)^{-1} \int d^{N_B} z \, \mathrm{e}^{-1} \, \epsilon^{\underline{a}_1 \cdots \underline{a}_{N_B}} f_{\underline{a}_1 \cdots \underline{a}_{N_B}} \quad , \tag{8}$$

and for Δ to truly correspond to a topological index, we must have $f_{\underline{a}_1 \cdots \underline{a}_{N_B}}$ satisfy a Bianchi-type identity (for the purely bosonic case this is trivial)

$$\mathrm{e}_{[\underline{a}_1|} f_{|\underline{a}_2 \cdots \underline{a}_{1+N_B}]} - c_{[\underline{a}_1 \underline{a}_2|}{}^{\underline{d}} f_{\underline{d}|\underline{a}_3 \cdots \underline{a}_{1+N_B}]} = 0 \quad , \tag{9}$$

and as well $f_{\underline{a}_1 \cdots \underline{a}_{N_B}}$ must <u>not</u> be globally defined over the entire N_B-dimensional bosonic manifold. In this equation $\mathrm{e}_{\underline{a}}$ denotes the local frame field operator of the bosonic sub-manifold and $c_{\underline{a}\underline{b}}{}^{\underline{c}}$ denotes the associated anholonomy.

What are the corresponding structures available over a Salam-Strathdee superspace?

In 1981 [12], I proposed the initial formulation of <u>irreducible</u> off-shell super p-forms for Salam-Strathdee superspace as a generalization of supersymmetric Yang-Mills theory. Although explicitly presented in a 4D, N = 1 superspace, the general structure is ubiquitous to all superspaces. The distinctive feature of the 1981 proposal was that it showed the constraints, Bianchi identities, and pre-potential solutions all exist for a simplex of super p-forms in exactly the same way as in supersymmetric Yang-Mills theory.

In 1983 [13], I was able to derive a further interesting result. If a super N_B-form like $F_{\underline{A}_1 \cdots \underline{A}_{N_B}}$ satisfies a set of Bianchi identities (i.e. it is super-closed), it follows that <u>independent</u> of its constraints and in the presence of supergravity, there exist a

W-Z gauge where

$$\Big(F_{\underline{a}_1 \cdots \underline{a}_{N_B}} | \Big) ~=~ \Big[~ \widetilde{f}_{\underline{a}_1 \cdots \underline{a}_{N_B}} ~+~ \lambda^{(N_B,1)} \psi_{[\underline{a}_1 |}{}^{\alpha_1} \Big(F_{\alpha_1 | \underline{a}_2 \cdots \underline{a}_{N_B}]} | \Big) \\ ~+~ \lambda^{(N_B,2)} \psi_{[\underline{a}_1 |}{}^{\alpha_1} \psi_{|\underline{a}_2 |}{}^{\alpha_2} \Big(F_{\alpha_1 \alpha_2 | \underline{a}_3 \cdots \underline{a}_{N_B}]} | \Big) \cdots \\ ~+~ \lambda^{(N_B,N_B)} [\, \psi_{\underline{a}_1}{}^{\alpha_1} \psi_{\underline{a}_2}{}^{\alpha_2} \cdots \psi_{\underline{a}_{N_B}}{}^{\alpha_{N_B}} \,] \Big(F_{\alpha_1 \alpha_2 \cdots \alpha_{N_B}} | \Big) ~ \Big] ~~~, \tag{10}$$

here $\widetilde{f}_{\underline{a}_1 \cdots \underline{a}_{N_B}}$ is an ordinary bosonic closed N_B-form, $\psi_{\underline{a}}{}^{\alpha}$ denotes the component gravitino field and $\lambda^{(N_B,i)}$ are a set of constants that are easily derivable. My original derivation of this was in the context of 4D, N = 4 supergravity but that derivation can easily be extended to all values of N_B and N_F.

Now the interesting thing about this equation is that I can isolate $\widetilde{f}_{\underline{a}_1 \cdots \underline{a}_{N_B}}$ (which can differ from $f_{\underline{a}_1 \cdots \underline{a}_{N_B}}$ by exact terms) to find

$$\widetilde{f}_{\underline{a}_1 \cdots \underline{a}_{N_B}} ~=~ \Big[~ \Big(F_{\underline{a}_1 \cdots \underline{a}_{N_B}} | \Big) ~-~ \lambda^{(N_B,1)} \psi_{[\underline{a}_1 |}{}^{\alpha_1} \Big(F_{\alpha_1 | \underline{a}_2 \cdots \underline{a}_{N_B}]} | \Big) \\ ~-~ \lambda^{(N_B,2)} \psi_{[\underline{a}_1 |}{}^{\alpha_1} \psi_{|\underline{a}_2 |}{}^{\alpha_2} \Big(F_{\alpha_1 \alpha_2 | \underline{a}_3 \cdots \underline{a}_{N_B}]} | \Big) \cdots \\ ~-~ \lambda^{(N_B,N_B)} [\, \psi_{\underline{a}_1}{}^{\alpha_1} \psi_{\underline{a}_2}{}^{\alpha_2} \cdots \psi_{\underline{a}_{N_B}}{}^{\alpha_{N_B}} \,] \Big(F_{\alpha_1 \alpha_2 \cdots \alpha_{N_B}} | \Big) ~ \Big] ~~~. \tag{11}$$

Upon multiplying by an ϵ-tensor and integrating $(1/N_B!) \int d^{N_B} z \, \mathrm{e}^{-1}$, I find

$$\widetilde{\Delta} ~=~ \widehat{\Delta} ~~~, \tag{12}$$

$$\widetilde{\Delta} ~\equiv~ (N_B!)^{-1} \int d^{N_B} z ~ \mathrm{e}^{-1} \, \epsilon^{\underline{a}_1 \cdots \underline{a}_{N_B}} \, \widetilde{f}_{\underline{a}_1 \cdots \underline{a}_{N_B}} ~~~, \tag{13}$$

$$\widehat{\Delta} \equiv ~ \int d^{N_B} z ~ \mathrm{e}^{-1} ~ \epsilon^{\underline{a}_1 \cdots \underline{a}_{N_B}} \Big[~ (N_B!)^{-1} \Big(F_{\underline{a}_1 \cdots \underline{a}_{N_B}} | \Big) ~-~ \lambda^{(N_B,1)} \psi_{\underline{a}_1}{}^{\alpha_1} \Big(F_{\alpha_1 \underline{a}_2 \cdots \underline{a}_{N_B}} | \Big) \\ ~-~ \lambda^{(N_B,2)} \psi_{\underline{a}_1}{}^{\alpha_1} \psi_{\underline{a}_2}{}^{\alpha_2} \Big(F_{\alpha_1 \alpha_2 \underline{a}_3 \cdots \underline{a}_{N_B}} | \Big) \cdots \\ ~-~ \lambda^{(N_B,N_B)} (N_B!)^{-1} \, [\, \psi_{\underline{a}_1}{}^{\alpha_1} \psi_{\underline{a}_2}{}^{\alpha_2} \cdots \psi_{\underline{a}_{N_B}}{}^{\alpha_{N_B}} \,] \Big(F_{\alpha_1 \alpha_2 \cdots \alpha_{N_B}} | \Big) ~ \Big] ~~~. \tag{14}$$

I now define the supertopological index $\widehat{\Delta}$ that was introduced into the diagram by asserting that equation (14) *is* the correct definition of how to integrate the closed super N_B-form $F_{\underline{A}_1 \cdots \underline{A}_{N_B}}$ over the entirety of the superspace[6]! Since $\widetilde{f}_{\underline{a}_1 \, \cdots \, \underline{a}_{N_B}}$ typically differs from $f_{\underline{a}_1 \, \cdots \, \underline{a}_{N_B}}$ by exact terms we have

$$\widehat{\Delta} ~=~ \Delta + \cdots ~~~~. \tag{15}$$

[6]For a previous proposal to define the integration theory of closed super p-forms, see the work of ref. [14].

So the definition above certainly enforces the Ethereal Conjecture but how does this solve the problem of finding $\mathcal{D}^{N_F}$?

The answer lies in the fact that the field strengths superfields in $\widehat{\Delta}$ (i.e. the F's) must be chosen to be subject to the constaints implied by irreducibility of the super N_B-form. In this case a number of the F's vanish and the remaining ones, via the solution of their Bianchi identities, are related by ∇_α, the spinorial derivative. When this solution for the various components of F is inserted into $\widehat{\Delta}$, as if by magic the operator $\mathcal{D}^{N_F}$ appears in all the cases I have studied. Let me show by some explicit examples how this topological tool works.

The simplest of all 2D supergravity theories is (1,0) or heterotic supergravity [15] which is described by a set of covariant derivatives (∇_+, $\nabla_=$, $\nabla_{\ddagger}$) satisfying the commutator algebra and single differential equation below

$$
\begin{aligned}
&[\, \nabla_+ \, , \, \nabla_+ \, \} = i2\nabla_{\ddagger} \quad , \quad [\, \nabla_+ \, , \, \nabla_{\ddagger} \, \} = 0 \quad , \quad \nabla_+ \Sigma^+ = \tfrac{1}{2}\mathcal{R} \quad , \\
&[\, \nabla_+ \, , \, \nabla_= \, \} = -i2\Sigma^+ \mathcal{M} \quad , \quad [\, \nabla_{\ddagger} \, , \, \nabla_= \, \} = -(\, \Sigma^+ \nabla_+ \; + \; \mathcal{R}\mathcal{M}) \quad .
\end{aligned} \tag{16}
$$

The quantities Σ^+ and $\mathcal{R}$ are field strength superfields and $\mathcal{M}$ denotes the generator of the 2D Lorentz group defined to act according to the rules; $[\mathcal{M}, \psi_+] = \frac{1}{2}\psi_+$, $[\mathcal{M}, \psi_-] = -\frac{1}{2}\psi_-$, $[\mathcal{M}, \mathrm{e}_{\ddagger}] = \mathrm{e}_{\ddagger}$ and $[\mathcal{M}, \mathrm{e}_=] = -\mathrm{e}_=$. On defining $\Sigma^+ \,|$ as the limit of Σ^+ as the Grassmann coordinate is taken to zero and similarly for $\mathcal{R}\,|$, we find

$$
\Sigma^+ \,| = \; -\psi_{\ddagger, \; =}{}^{+} \; = \; -[\; \mathrm{e}_{\ddagger} \psi_={}^{+} - \mathrm{e}_= \psi_{\ddagger}{}^{+} - c_{\ddagger, \; =}{}^{\ddagger} \psi_{\ddagger}{}^{+} - c_{\ddagger, \; =}{}^{=} \psi_={}^{+} \;] \; , \tag{17}
$$

$$
\begin{aligned}
r_{\ddagger, \; =}(\omega) \;\; &= \;\; -[\; \mathrm{e}_{\ddagger}\omega_= - \mathrm{e}_=\omega_{\ddagger} - c_{\ddagger, \; =}{}^{\ddagger}\omega_{\ddagger} - c_{\ddagger, \; =}{}^{=}\omega_= \;] \; , \\
\nabla_+ \Sigma^+ \,| &= \; -\tfrac{1}{2}[\; r_{\ddagger, \; =}(\omega) + i2\psi_{\ddagger}{}^{+}\psi_{\ddagger, \; =}{}^{+} \;] \quad ,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&\widehat{\nabla}_{\ddagger} \equiv \mathrm{e}_{\ddagger} + \omega_{\ddagger}\mathcal{M} \quad , \quad \widehat{\nabla}_= \equiv \mathrm{e}_= + \omega_=\mathcal{M} \quad , \quad \omega_{\ddagger} \; = \; c_{\ddagger, \; =}{}^{=} \quad , \\
&\omega_= \; = \; c_{\ddagger, \; =}{}^{\ddagger} + i2\psi_{\ddagger}{}^{+}\psi_={}^{+} \quad , \quad \mathrm{e}_a \equiv \mathrm{e}_a{}^m \partial_m \quad , \quad [\mathrm{e}_a, \mathrm{e}_b] = c_{a,b}{}^c \mathrm{e}_c \quad .
\end{aligned} \tag{19}
$$

But $\mathcal{R}$ is the vector-vector component of the super 2-form R_{AB}. Thus, we take the first equality in (17) and use it to replace the $\psi_{\ddagger, \; =}{}^{+}$ term on the last line of (18) to find,

$$
-\tfrac{1}{2} \; r_{\ddagger, \; =}(\omega) \;\; = \;\; \Big[\; \Big(\; \nabla_+ \; - \; i\psi_{\ddagger}{}^{+} \Big) \Sigma^+ \,| \; \Big] \quad . \tag{20}
$$

This is a special case of (11) and following the general discussion we enforce the E.C. by defining

$$
\widetilde{\Delta} \;\; \equiv \;\; -\tfrac{1}{2} \int d^2\sigma \; \mathrm{e}^{-1} \, r_{\ddagger, \; =}(\omega(e, \psi)) \quad , \tag{21}
$$

$$
\widehat{\Delta} \;\; \equiv \;\; \int d^2\sigma \; \mathrm{e}^{-1} \Big[\; \Big(\; \nabla_+ \; - \; i\psi_{\ddagger}{}^{+} \Big) \Sigma^+ \,| \; \Big] \quad . \tag{22}
$$

and according to the E.I.T. and E.C. it must also be the case that

$$\begin{aligned}\int d^2\sigma\, d\zeta^- \ \mathrm{E}^{-1}\, \mathcal{L}_- \equiv & \int d^2\sigma \ \mathrm{e}^{-1} \Big[\ \mathcal{D}_+ \mathcal{L}_- \,| \ \Big] \\ = & \int d^2\sigma \ \mathrm{e}^{-1} \Big[\ \Big(\ \nabla_+ \ - \ i\psi_{\ddagger}{}^+ \Big) \mathcal{L}_- \,| \ \Big] \quad . \end{aligned} \tag{23}$$

exactly as stated in the first work of reference [15].

Now the expression for $\widetilde{\Delta}$ in (21) allows us to calculate the form of the terms in $\widetilde{\Delta} \ = \ \Delta \ + \ ....$ This is done by observing that

$$r_{\ddagger, \ =}(\omega(e, \psi)) \ = \ r_{\ddagger, \ =}(\omega(e, 0)) \ + \ i2 \{ \nabla_{\ddagger}(e) [\ \psi_{\ddagger}{}^+ \, \psi_{=}{}^+ \ ] \, \} \quad . \tag{24}$$

so that

$$\widetilde{\Delta} \ \equiv \ -\tfrac{1}{2} \ \int d^2\sigma \ \mathrm{e}^{-1} \, r_{\ddagger, \ =}(\omega(e, 0)) \ - \ i \int d^2\sigma \ \mathrm{e}^{-1} \, \{ \, \partial_m [\, \mathrm{e}_{\ddagger}{}^m (\ \psi_{\ddagger}{}^+ \, \psi_{=}{}^+ \ ) \,] \, \} \quad . \tag{25}$$

We see that the first term above is $\Delta \ = \ 2\pi(g - 1)$ (where g is the genus of the manifold), the usual topological index on a 2-manifold,

$$\Delta \ \equiv \ -\tfrac{1}{2} \ \int d^2\sigma \ \mathrm{e}^{-1} \, r_{\ddagger, \ =}(\omega(e, 0)) \quad , \tag{26}$$

and the second term in (25) is what was indicated by ... in the E.C.

Perhaps the reader was not impressed by the (1,0) example. So let's repeat all of this in the more complicated case of 3D, N = 1 superspace. In 1979 [16] the superspace description of 3D, N = 1 irreducible off-shell supergravity was first given

$$\begin{aligned} [\nabla_\alpha \ , \ \nabla_\beta\} &= i2(\gamma^c)_{\alpha\beta} \, [\nabla_c - R\mathcal{M}_c] \quad , \\ [\nabla_\alpha \ , \ \nabla_b\} &= i(\gamma_b)_\alpha{}^\delta [\ \tfrac{1}{2} R \nabla_\delta \ + \ (\Sigma_\delta{}^d \ + \ i\tfrac{2}{3}(\gamma^d)_\delta{}^\epsilon (\nabla_\epsilon R))\mathcal{M}_d \ ] \\ &\quad + (\nabla_\alpha R)\mathcal{M}_b \quad , \\ [\nabla_a \ , \ \nabla_b\} &= -\tfrac{1}{2}\epsilon_{abc} [\ \Sigma^{\alpha c} + i\tfrac{2}{3}(\gamma^c)^{\alpha\beta}(\nabla_\beta R) \ ] \nabla_\alpha \\ &\quad - \epsilon_{abc} [\ \mathcal{R}^{cd} \ + \ \tfrac{2}{3}\eta^{cd}(\nabla^2 R \ - \ \tfrac{3}{2}R^2) \ ] \mathcal{M}_d \quad , \end{aligned} \tag{26}$$

where $\mathcal{R}^{ab} - \mathcal{R}^{ba} = \eta_{ab}\mathcal{R}^{ab} = (\gamma_d)^{\alpha\beta}\Sigma_\beta{}^d = 0$ and

$$\nabla_\alpha \Sigma_\beta{}^c = i(\gamma_b)_{\alpha\beta}\mathcal{R}^{bc} \ - \ \tfrac{2}{3} [\ C_{\alpha\beta}\eta^{cd} \ + \ i\tfrac{1}{2}(\gamma_b)_{\alpha\beta}\epsilon^{bcd} \ ] \, (\nabla_d R) \quad . \tag{27}$$

In writing these results, their form was simplified by replacing the usual Lorentz generator according to: $\mathcal{M}_{bc} \to \epsilon_{bc}{}^a \mathcal{M}_a$, so that when acting on a spinor ψ_α or a vector v_a we have

$$[\, \mathcal{M}_a \ , \ \psi_\alpha \,] \ = \ i\tfrac{1}{2}(\gamma_a)_\alpha{}^\beta \psi_\beta \quad , \quad [\, \mathcal{M}_a \ , \ v_b \,] \ = \ \epsilon_{ab}{}^c v_c \quad . \tag{28}$$

Now R is a component of a super 2-form, but in 3D we need a super 3-form. Using the formalism of the 1981 work [12], it is easy to show that an irreducible 3D, N = 1 closed super 3-form is described by $\mathcal{G}_{ABC}$ where

$$\mathcal{G}_{\alpha\beta\gamma} ~=~ 0 ~~~,~~~ \mathcal{G}_{\alpha\beta c} ~=~ i2(\gamma_c)_{\alpha\beta}\mathcal{G} ~~~,$$
$$\mathcal{G}_{\alpha bc} ~=~ i\epsilon_{abc}(\gamma^a)_\alpha{}^\beta\,(\nabla_\beta\mathcal{G}\,) ~~~,~~~ \mathcal{G}_{abc} ~=~ \epsilon_{abc}\,[\,\nabla^2\mathcal{G} ~-~ R\mathcal{G}\,] ~~~~~. \tag{29}$$

For this theory, the general result in (11) takes the form

$$\mathcal{G}_{abc}| ~=~ \widetilde{g}_{abc} ~+~ \epsilon_{abc}\Big[\,i\psi_d{}^\alpha(\gamma^d)_\alpha{}^\beta(\,\nabla_\beta\mathcal{G}|\,) ~-~ i\epsilon^{def}\psi_d{}^\alpha(\gamma_e)_{\alpha\beta}\psi_f{}^\beta(\,\mathcal{G}|\,)\,\Big] ~~~~, \tag{30}$$

so that after substitution of the last result from (29) into the lhs of (30) it follows that

$$\tfrac{1}{6}\epsilon^{abc}\widetilde{g}_{abc} ~=~ (\mathcal{D}^2\mathcal{G}|\,) ~~~, \tag{31}$$

where the explicit form of the operator $\mathcal{D}^2$ is given by

$$\mathcal{D}^2 ~\equiv~ \nabla^2 ~-~ i\psi_a{}^\alpha(\gamma^a)_\alpha{}^\beta\nabla_\beta ~-~ R ~+~ i\epsilon^{abc}\psi_a{}^\alpha(\gamma_b)_{\alpha\beta}\psi_c{}^\beta ~~~~. \tag{32}$$

Therefore I am to define

$$\widetilde{\Delta} ~\equiv~ \frac{1}{6}\int d^3x~\mathrm{e}^{-1}\,\epsilon^{abc}\widetilde{g}_{abc} ~~~, \tag{33}$$

$$\widehat{\Delta} ~\equiv~ \int d^3x~\mathrm{e}^{-1}\Big(\,\mathcal{D}^2\mathcal{G}\,|\,\Big) ~~~, \tag{34}$$

and again using the E.I.T. and E.C. to define

$$\int d^3x~d^2\theta\,\mathrm{E}^{-1}\,\mathcal{L} ~\equiv~ \int d^3x~\mathrm{e}^{-1}\Big[\,\mathcal{D}^2\mathcal{L}\,|\,\Big] ~~~~. \tag{35}$$

Since $\mathcal{D}^2$ was derived via the E.C., we might want to check it on another choice of $\mathcal{L}$ such as $\mathcal{L} ~=~ R$. It is known for this choice that $S_{SG} ~\propto~ \int d^3x~d^2\theta\mathrm{E}^{-1}R$ is the correct answer. After the usual projection techniques, I find

$$\int d^3xd^2\theta\,\mathrm{E}^{-1}\,R = \int d^3x\mathrm{e}^{-1}\,[\,-\tfrac{1}{2}\epsilon^{abc}\,(\mathcal{R}_{abc}(\omega) ~+~ \psi_{a\alpha}\Psi_{bc}{}^\alpha) ~-~ B^2\,] ~~~. \tag{36}$$

where the $\Psi_{ab}{}^\beta$ is the usual component level gravitino field strength and the spin-connection is given by,

$$\omega_a{}^b ~=~ \tfrac{1}{4}\epsilon^{bcd}\,[\,C_{cda} ~-~ 2C_{acd} ~+~ i4\left(\psi_c{}^\alpha(\gamma_a)_{\alpha\beta}\psi_d{}^\beta ~+~ \psi_a{}^\alpha(\gamma_c)_{\alpha\beta}\psi_d{}^\beta\right)\,] ~-~ \tfrac{1}{2}B\delta_a{}^b ~~~. \tag{37}$$

Finally I have checked this same procedure using old minimal off-shell supergravity in 4D, N = 1 Wess-Zumino superspace to calculate the topological index $\widehat{\Delta}$ associated

with the 4D, N = 1 super 4-form multiplet described in the 1981 paper [12]. I find the result

$$\widehat{\Delta} \;=\; \int d^4x \; \mathrm{e}^{-1} \Big[\; - i \, (\mathcal{D}^2 \mathcal{F} \, |) \; + \; \mathrm{h.\,c.} \, \Big] \quad , \tag{38}$$

here the operator $\mathcal{D}^2$ is defined by

$$\mathcal{D}^2 \;\equiv\; \nabla^2 \;+\; i \overline{\psi}{}^{\underline{a}}{}_{\dot\alpha} \nabla_\alpha \;+\; 3 \overline{R} \;+\; \tfrac{1}{2} C^{\alpha\beta} \overline{\psi}{}_{\underline{a}}{}^{(\dot\alpha} \, \overline{\psi}{}_{\underline{b}}{}^{\dot\beta)} \quad , \tag{39}$$

and $\mathcal{F}$ is the lowest non-trivial super 4-form field strength component

$$F_{\alpha \, \beta \, \underline{c} \, \underline{d}} \;=\; C_{\dot\gamma\dot\delta} C_{\alpha(\gamma} C_{\delta)\beta} \overline{\mathcal{F}} \quad . \tag{40}$$

I note that the form of (38) suggests the formula

$$\int d\mu \; \mathcal{L}_{Gen} \;=\; \tfrac{1}{2} \int d\mu_c \; \mathcal{L}_c \;+\; \mathrm{h.\,c.} \quad . \tag{41}$$

So that the E.C. and E.I.T. imply

$$\int d\mu_c \; \mathcal{L}_c \;=\; \int d^4x \, \mathrm{e}^{-1} \, \Big[\mathcal{D}^2 \mathcal{L}_c \, | \, \Big] \quad , \tag{42}$$

acting on a chiral superfield (such as $\mathcal{F}$). More generally

$$\mathcal{L}_c \;=\; (\overline{\nabla}^2 \;+\; R) \, \mathcal{L}_{Gen} \quad , \tag{43}$$

so that we may define

$$\begin{aligned} \int d\mu \; \mathcal{L}_{Gen} = \; & \tfrac{1}{2} \Big\{ \; \int d^4x \, \mathrm{e}^{-1} \, \Big[\mathcal{D}^2 \; (\overline{\nabla}^2 \;+\; R) \, \mathcal{L}_{Gen} \, | \, \Big] \;+\; \mathrm{h.c.} \; \Big\} \\ \equiv \; & \int d^4x \, \mathrm{e}^{-1} \, \Big[\mathcal{D}^4 \; \mathcal{L}_{Gen} \, | \, \Big] \quad , \end{aligned} \tag{44}$$

where the operator $\mathcal{D}^4$ is defined by

$$\mathcal{D}^4 \;=\; \tfrac{1}{2} \Big[\; \mathcal{D}^2 \;\; (\overline{\nabla}^2 \;+\; R) \;+\; \overline{\mathcal{D}}^2 \;\; (\nabla^2 \;+\; \overline{R}) \, \Big] \quad . \tag{45}$$

This final result can be seen to coincide exactly with the result in our book '*Superspace*' where it was 'derived by a handicraft' argument.

Thus, I see that there is excellent support for the E.I.T. and E.C. from a number of explicit cases. I have also found numerous other examples. I am still checking even more examples in an attempt to understand if there are any limitations on this method.

I think the E.C. is a universal feature of all supergravity theories that has escaped our notice since the beginning of the era of using Wess-Zumino superspace! I also

have some evidence that the E. C. plays an even more important role than I presented here. In some examples I know, there occur topological obstructions to the imposition of the E. C. The most interesting point about these obstructions is that they take the form of the supergravity constraints themselves! It is perhaps not too optimistic to hope that at last we have begun to grasp the 'deep' reason why constraints must be imposed in supersymmetrical theories. The answer seems to be to enforce the E. C.

I further conjecture that the E.C. will ultimately be found to apply to even covariant string field theory! The reasoning goes as follows. In a fully covariant and geometrical approach to string and superstring field theory, one must be confronted with calculating the integral $\int d^D\mathbf{X}(\sigma)\, d\mathbf{B}(\sigma)\, d\mathbf{C}(\sigma)$ (here we consider the bosonic string for the sake of simplicity). It ought to be possible to write an equation like

$$\int d^D\mathbf{X}(\sigma)\ d\mathbf{B}(\sigma)\ d\mathbf{C}(\sigma)\ =\ \int d^D\mathbf{X}(0)\ \mathcal{D}^{(\infty)} \quad , \tag{46}$$

so that the string coordinate zero-modes define the manifold of an ordinary appearing field theory. The oscillator modes (of all types) define the ectoplasm of the string space. Thus in a fully geometrical approach to covariant string field theory I expect that there should exist an operator $\mathcal{D}^{(\infty)}$ that appears in a 'stringy' E.I.T.

If this conjecture proves to be true, it provides an elegantly simple basis for understanding why even if we live in a universe described by fiber bundles, Kaluza-Klein spaces, Wess-Zumino superspace, strings, superstrings, heterotic strings, branes, M-theory, F-theory etc., the topological triviality of all the extra "coordinates" may forbid their having direct physical consequences (at least in the absence of strong coupling given the current views of strong/weak duality). The Ethereal Conjecture, properly interpreted, may be a physical principle.

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